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GENERATION OF LONG WAVES BY THE BOTTOM CONTOUR IN A TWO-LAYER FLOW

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1. INTRODUCTION

The influence of bottom irregularities on horizontal flow of a stratified fluid can manifest itself through essentially nonlinear effects due to the propagation of long upstream waves and a rearrangement of the flow, under certain conditions including partial or complete blocking of the entire region of flow by an obstacle [1, 2]. Despite the complicated wave pattern in the vicinity of the obstacle, in a number of cases the salient features of the flow can be described by the equations of multilayer shallow water. For two-layer and multilayer shallow water, respectively, Baines [3, 4] gave a detailed theoretical description of possible flow regimes, a comparison of his experiments on the motion of an obstacle in a two-layer liquid a rest, and a classification of waves propagating upstream and flows over an obstacle as a function of the mainstream parameters and the barrier height. This problem was studied numerically in [5]. In comparison with the analogous problem of the motion of an obstacle in a one-layer liquid [1, 6] not only does the wave pattern become more complicated but also basically new flow regimes appear. In steady flow regimes found experimentally in [1, 3] critical and supercritical flows in front of the obstacle become a supercritical flow behind it, remaining supercritical everywhere above the barrier. Above the obstacle and immediately beyond it there may be hydraulic jumps and whirls, converting supercritical flow into supercritical flow once again. Externally this situation contradicts the concept of control section, generally accepted in hydraulics, within which during steady flow various states in front of and beyond the obstacle can be realized only if the flow is subcritical in front of the barrier, critical above the crest, and supercritical behind it.

Our aim here, within the shallow water theory, is to make a theoretical study of possible flow regimes of a two-layer liquid in the vicinity of an obstacle moving with constant velocity and, in particular, to substantiate the possibility of a forced regime of flow past the barrier, during which supercritical flow occurs over the crest. Another feature is that the classification of flows, taking into account the initial shift of velocity in the layers, is more complete than in the studies mentioned above. In particular, for flows with a nonzero initial shift of velocity a supercritical flow regime was found; the regime contains a pair consisting of a hydraulic jump and a well and is characterized by an abrupt change in the amplitudes of the waves for small changes in the flow parameters. The last circumstance is a source of hysteresis that is observed in numerical calculations when the flow reaches one steady-state regime or another, depending on how the immersed body is accelerated in the flow. The cause of the hysteresis is different from the analogous phenomenon noted in [3]. There, the overlap of the regions in which various kinds of streaming (supercritical and with hydraulic jump gone) occur is attributed to the application of different laws of conservation for the flow of the liquid over an even bottom and over a barrier. In this study, in much the same way as in [7], a single system is used to describe two-layer potential flow over an uneven bottom and the choice of laws of conservation, which is in itself problematical for this class of flows, is validated in Section 2. Hysteresis when the flow reaches the vicinity of the body in various steady-state regimes at the same towing speeds is due primarily to the nonmonotonic dependence of the parameters of the mainstream over an obstacle on the height of the obstacle and manifests itself during forced streaming regimes.

2. Equations of Two-Layer Shallow Water. In the long-wave approximation the equations of two-layer shallow water are derived from the assumptions of hydrostatic pressure and a uniform depth distribution of the density and horizontal component of the velocity from the equations of motion of an ideal inhomogeneous liquid [8].

In the Boussinesq approximation for $(\rho^- - \rho^+)/\rho_0 << 1$ the equations of a plane-parallel two-layer flow of liquid under a horizontal cover have the form

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$$h_t^+ + (h^+u^+)_x = 0,\ h_t^- + (h^-u^-)_x = 0,\ u_t^+ + u^+u_x^+ + \frac{1}{\rho_0}p_x^+ = 0,$$
$$u_t^- + u^-u_x^- + b(h_x^- + z_x) + \frac{1}{\rho_0}p_x^+ = 0,\ \ h^+ + h^- + z = H \equiv \text{const.} \tag{2.1}$$

Here h^+ and h^- are the thicknesses and u^+ and u^-, the horizontal velocities, of the upper and lower layers; $p^+ = p$ is the pressure on the upper cover; H is the total channel depth; the equation $z = z(t, x)$ specifies the position of the channel bottom (Fig. 1); $b = (\rho^- - \rho^+)g/\rho_0$ reflects the effect of buoyancy on the dynamics of the layers of the liquid.

The system (2.1) is essentially nonlinear. Even from the initially smooth distribution of the layer thicknesses and velocities, therefore, in time solutions are worked out with large gradients of these quantities and the hypotheses of a hydrostatic pressure distribution and a smooth variation of the flow parameters become meaningless. The resulting discontinuous solutions describe such phenomena in a real liquid as internal hydraulic jumps and surges. In the two-layer model (2.1) the relations necessary to determine the position and amplitude of the jump cannot be obtained at discontinuities, even though this system has an infinite number of laws of conservation that uniquely determine the relations at discontinuities. The approach most employed by researchers is that proposed in [9]. It consists of obtaining relations at a jump on the basis of assumptions of a hydrostatic pressure distribution at the interface between the layers inside the jump. One of the disadvantages of this approach is that a generalized (discontinuous) solution of the system (2.1) cannot be found without isolating the discontinuity lines since no set of laws of conservation correspond to the conditions at a discontinuity. Moreover, under certain conditions the relations obtained lead to a nonphysical energy redistribution in the moving layers. Another approach was presented in [10]. The laws of conservation of mass, total momentum, and energy in one layer are used as the basic set of laws of conservation. The asymmetry in the choice of laws of conservation is based on the fact that under certain conditions (e.g., one layer much thinner than the other) the region of mixing that arises in the jump can extend to the bottom and, therefore, the energy losses in the jump apply to such a layer. A review of experimental and theoretical studies to analyze the structure of waves in a two-layer flow that arises during streamline flow past an obstacle can be found in [6].

The principal contradiction in the model of two-layer flow because the laws of conservation of mass, moment, and energy cannot be satisfied simultaneously is resolved by going over to a three-layer model [11]. The third layer is an interlayer formed as a result of mixing and intensive wave motion at the interface between homogeneous layers. Analysis of the solution of the system consisting of the laws of conservation of mass in each layer, energy in the top and bottom homogeneous layers, and total momentum and total energy (including the small-scale component of the kinetic energy in the interlayer) gives a real picture of the motion of the layers with allowance for mixing. In particular, this model describes the conditions of the transition of the mixing layer into a submerged flow and the system of equations reduces to the system of laws of conservation used in [10]. The complete system of equations for a three-layer flow is difficult to analyze because of the large number of nonlinear equations as well as the inhomogeneity of the equations as a result of the inclusion of turbulent entrainment of liquid from the homogeneous layers into the interlayer. For a complete picture, therefore, it is necessary to study the simplest limiting cases.

The equations obtained for two-layer when the interlayer thickness tends to zero constitute one such limit. The system of equations in this case consists of the laws of conservation of mass and energy in homogeneous layers. And although it must be considered to be approximate, just as the system mentioned above, it does substantially simplify analysis of the wave pattern and, as is shown below, describes the experimentally observed flow regimes in the problem of streamline flow of a steady-state two-layer liquid stream past an obstacle. In Section 3 we give the main properties of self-similar solutions of such a system, which was studied in [12] on the example of the problem of the decay of an arbitrary discontinuity.

3. Problem of the Decay of an Arbitrary Discontinuity. It is convenient to go over to the dimensionless variables $h^\pm \to h^\pm/H$, $z \to z/H$, $u^\pm \to u^\pm/\sqrt{bH}$, and $p \to p/\rho_0 bH$, i.e., to set $H = 1$, $b = 1$, and $\rho_0 = 1$ in (2.1). In a channel with an even bottom ($z(t, x) \equiv 0$) the flow rate $Q = h^+u^+ + h^-u^-$ does not depend on x. without loss of generality we can obtain $Q \equiv 0$ by going over to the appropriate coordinate system. For the variables $h = h^-$ and $\gamma = u^- - u^+$ the system (2.1) takes on the form ($0 \le h \le 1$)

$$h_t + (h(1 - h)\gamma)_x = 0,\ \gamma_t + \left(\frac{1}{2}(1 - 2h)\gamma^2 + h\right)_x = 0. \tag{3.1}$$

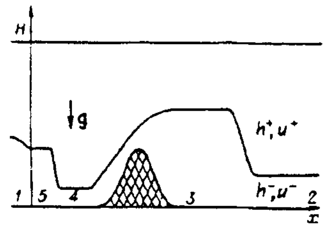

Fig. 1

System (3.1) is written in divergent form, which makes it possible to determine motion with discontinuities. The pertinent laws of conservation are the laws of conservation of mass of each layer and of the potentiality of flow upon transition from the region with abrupt changes of quantities in the flow. As mentioned above, conservation of the total momentum is ensured by a rearrangement of the flow in the neighborhood of the interlayer, whose thickness is assumed to be small and is not taken into account in the given model. The structure of the discontinuous solutions for (3.1) can be considered on the example of the Cauchy problem with stepped initial data (the problem of the decay of an arbitrary discontinuity)

$$(h(0, x), \gamma(0, x)) = \begin{cases} (h_1, \gamma_1), & x < 0, \\ (h_2, \gamma_2), & x > 0. \end{cases} \tag{3.2}$$

The solvability of the problem (3.1), (3.2) was studied in [12], where in particular it was proved that for the initial data from the region of hyperbolicity $\Omega = \{(h, \gamma)\colon 0 < h < 1, |\gamma| < 1\}$ of the system (3.1) there exists a single solution of the problem (3.1), (3.2) with the values (h, γ) in Ω that contains a finite number of aligned simple waves and discontinuity lines. An aligned simple wave is taken to mean a continuous solution that depends on a combination of variables $\zeta = x/t$. Additional stability conditions, ensuring the uniqueness and continuous dependence of the solutions on the initial data, are satisfied on discontinuity lines of the first kind. Since in much the same way as equations of one-layer shallow water and the equations of gas dynamics the problem of the decay of an arbitrary discontinuity can give an idea of the salient features of the solutions of the Cauchy problem with arbitrary initial data, it makes sense to consider it in greater detail.

The equations of the characteristics of the system (3.1)

$$dx/dt = \lambda^{\pm} = (1 - 2h)\gamma \pm \sqrt{h(1 - h)(1 - \gamma^2)}$$

show that this system is hyperbolic type for $(h, \gamma) \in \Omega$. In Riemann invariants

$$S = (1 - 2h) - 2\sqrt{h(1 - h)(1 - \gamma^2)},$$
$$R = (1 - 2h) + 2\sqrt{h(1 - h)(1 - \gamma^2)}$$

Eqs. (3.1) take on the form

$$S_t + \lambda^- S_x = 0, \quad R_t + \lambda^+ R_x = 0. \tag{3.3}$$

In the plane (h, γ) of the hodography by virtue of (3.3) the characteristics are mapped by ellipses Λ_r, which are specified by

$$(\gamma + r(2h - 1))^2 + (1 - r^2)(2h - 1)^2 = 1 - r^2 \; (r^2 < 1)$$

and inscribed in the hyperbolicity rectangle Ω (Fig. 2), the parts with positive slope corresponding to the S wave ($S \equiv$ const) and with a negative slope, to the R wave ($R \equiv$ const). From the laws of conservation (3.1) it follows that the relations

$$D_s[h] = [h(1 - h)\gamma], \quad D_s[\gamma] = \left[\frac{1}{2}(1 - 2h)\gamma^2 + h\right] \tag{3.4}$$

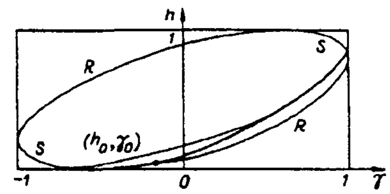

Fig. 2

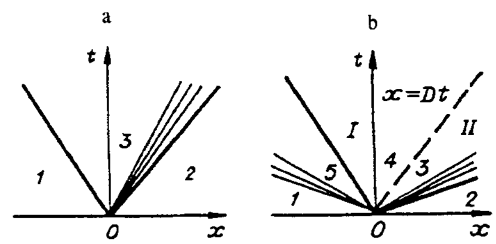

Fig. 3

are satisfied on discontinuity lines with the equation $D_s = dx/dt$. Here $[f] = f(t, x(t) - 0) - f(t, x(t) + 0)$. The discontinuity propagates to the left (first kind), if $[h][\gamma] < 0$ and to the right (second kind) if $[h][\gamma] > 0$. Equations (3.4) can be written as

$$\frac{\gamma - \gamma_0}{\sqrt{1 - \gamma\gamma_0}} = \pm \frac{h - h_0}{\sqrt{\frac{1}{2}h + \frac{1}{2}h_0 - hh_0}}. \tag{3.5}$$

The relations $\gamma = \gamma^{\pm}(h; h_0, \gamma_0)$ and $D_s = \sigma(h, \gamma; h_0, \gamma_0)$ are found in explicit form from (3.4) and (3.5). If (h_0, γ_0) is the state in front of the wave, then the function $\gamma = \gamma^{-}(h; h_0, \gamma_0)$ decreases with increasing h for discontinuities of the first kind while $\gamma = \gamma^{\pm}(h; h_0, \gamma_0)$ increases with h for discontinuities of the second kind. The jump with values (h^+, γ^+) and (h^-, γ^-) to the right and left of the discontinuity is stable upon satisfaction of the inequalities

$$D_s = \sigma(h^+, \gamma^+; h^-, \gamma^-) \leq \sigma(h, \gamma^-(h; h^-, \gamma^-)),$$
$$D_s = \sigma(h^+, \gamma^+; h^-, \gamma^-) \geq \sigma(h, \gamma^+(h; h^+, \gamma^+)) \tag{3.6}$$

for discontinuities of the first and second kind, respectively, for $(h^+ - h)(h^- - h) \leq 0$. In contrast to studies (e.g., [3]) where the stability of the discontinuities was attributed to a decrease in the total energy in the jump, condition (3.6) does not require use of an additional law of conservation and expresses the monotonic growth of the velocity of the discontinuity as a function of its amplitude.

The self-similar solution of the problem (3.1), (3.2) can be found much like the gas dynamics equations [13] from the intersection of the wave adiabatic curves in Ω. An adiabatic curves of the first or second kind is taken to be a smooth curve on the (h, γ) plane with the equation $\gamma = w^{\pm}(h; h_0, \gamma_0)$ passing through the point (h_0, γ_0) and consisting of segments of simple R or S waves and stable shock transitions of the respective kinds. In [12] we showed that a single monotonically decreasing adiabatic curve of the first kind and a single monotonically increasing adiabatic curve of the second kind pass through each point (h_0, γ_0). The wave adiabatic curve consists of a shock transition segment, starting at (h_0, γ_0), and two segments of simple waves adjoining it on either side. The joining occurs at points (h, γ), where the condition $d\sigma/dh = 0$ is satisfied. At those points the velocity of the characteristics of the corresponding family $\lambda \pm (h, \gamma)$ agrees with the velocity $\delta(h, \gamma_0; h, \gamma)$ of the discontinuity and a simple wave adjoins the discontinuity smoothly (see Fig. 2). If the initial states (h_i, γ_i) $(i = 1, 2)$ in (3.2) belong to the same ellipse $\Lambda_r(r^2 < 1)$, then a wave adiabatic curve of the first kind passing through (h_1, γ_1) and a wave adiabatic curve of the second kind passing through (h_2, γ_2), as shown in [12], have a single point of intersection in Λ_r. The corresponding self-similar solution of the problem (3.1), (3.2), therefore, consists of two waves turned to theleft and the right and centered at (0, 0) (Fig. 3a). Each of these waves is either a combination consisting of a stable dis-

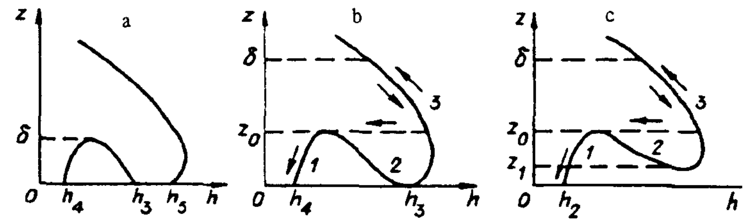

Fig. 4

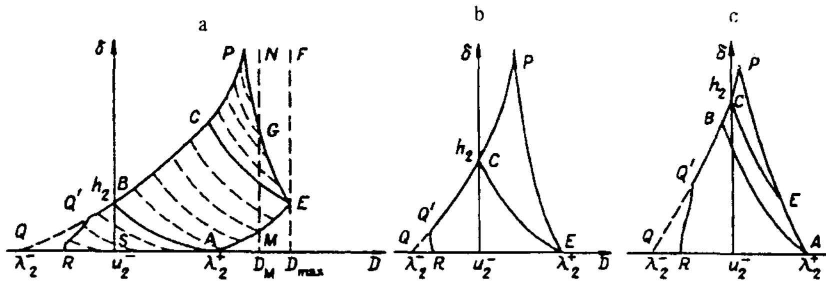

Fig. 5

continuity, propagating by the unperturbed state 1 or 2, and a following centered simple wave, or only a stable shock transition or centered wave.

If a value of $r(|r| \leq 1)$ such that $(h_1, \gamma_1) \in \Lambda_r$ for $i = 1, 2$ is not found but $(h_1, \gamma_1) \in \Omega$, then the solution of the problem (3.1), (3.2) does exist but goes beyond the limits of the square Ω; by analogy with the gas dynamics equations this denotes the appearance of "vacuum" regions in the solution, i.e., the degeneracy of the hyperbolicity. The analogy is reinforced by the circumstance that a change of variables reduces Eqs. (3.1) to the equations of a polytropic gas with an adiabatic exponent of 2 [8].

Because a maximum possible velocity $\sigma^{\pm}_{max}$ of propagation exists for the long-wave perturbations the salient features of the generation and propagation of waves in a two-layer system are qualitatively new compared to those in a one-layer liquid. These features are illustrated below with the example of the problem of streamline flow of a two-layer liquid past an obstacle.

4. Two-Layer Flow over an Obstacle. If the channel bottom is not flat $(z(t, x) \equiv 0)$, the system (2.1) does not have any self-similar solutions. If the external parameters characterizing the problem are fixed and the shape of the bottom and its rate of deformation are constant $(z = z_0(x - Dt), D \equiv \text{const})$, however, then the solution of (2.1) for long times becomes self-similar. A nearly steady-state regime is realized above the obstacle in this case. This circumstance can be used to formulate the problem of streamline flow of a two-layer liquid past an obstacle in the class of self-similar solutions.

Suppose that for $t = 0$ a short obstacle begins to move in a two-layer liquid along the channel bottom with constant velocity D. In front of and behind the obstacle the motion produces perturbations, which can be considered to be centered at the point (0, 0) in the plane (x, t) if the length of the obstacle is ignored in comparison with the scale of the waves. In this formulation we can use the structure of self-similar solutions, which was examined in Section 3, to construct the flow as a whole. In comparison with the problem (3.1), (3.2) only the number of possible configurations increases. The motion of the obstacle generates waves only when the states of the flow in front of and behind the obstacle are different. A discontinuity line $x = Dt$, corresponding to the trajectory of the motion of the body (Fig. 3b), that is new in comparison with the solution of (3.1), (3.2) appears in the (x, t) plane. The relations obtained at this discontinuity from an analysis of possible steady-state regimes of flow over an obstacle link the region $I(x < Dt)$ behind the body and the region II $(x > Dt)$ in front of it. The terms in front of and behind are arbitrary, as yet, since states 1 and 2 are equally justified. Regions I and II, however, are not equally justified. As is shown below, the flow in one of the regions, e.g., in region II, can be determined (independently of state 1) from an analysis of the conditions at the discontinuity $x = Dt$ for some parameters of the problem. The solution so obtained uniquely determines the state beyond the discontinuity $x = Dt$ and then the solution is found in region I. This is possible when for the length $x = Dt$ characteristics of the first kind come from region II and characteristics of the second kind

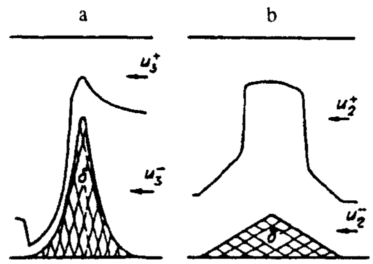

Fig. 6

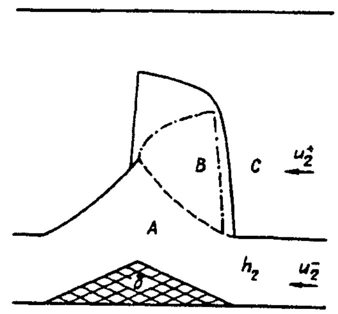

Fig. 7

from this line are in region II, i.e., in II we solve a mixed problem in which the state in front of the obstacle is subcritical relative to it. In region I the state behind the obstacle is supercritical, i.e., the characteristics of both kinds emerge from the line x = Dt into that region and, therefore, a Cauchy problem that is completely analogous to the problem (3.1), (3.2) is solved in I.

The direction of the motion of the body in the two-layer liquid must be determined in order to ascertain in which region (I or II) the streamline past the obstacle is subcritical. The region is said to be upstream if the velocity of the liquid in the layer in which the body is immersed (in this case, the lower layer) is lower than the velocity of the body. Then the motion of the body in the two-layer liquid controls the upstream flow if a subcritical regime is created in front of the obstacle and a supercritical regime behind it. The condition that ensures upstream control is the possibility of a steady-state flow of a two-layer liquid exiting relative to the obstacle and effecting a subcritical–supercritical conversion. To find the additional relations that ensure such a transition, we consider steady-state streamline flow past an obstacle in a coordinate system that moves along with the body.

5. Steady-State Flow over an Obstacle. Let us assume for definiteness that the body moves to the right, i.e., $u^-_c <$ D, where the subscript c indicates the state of the flow over the crest of the obstacle ($z'(x, c) = 0$, Fig. 1). Equation (2.1) gives the following integrals for steady-state streamline flow:

$$\begin{gathered} h(u^- - D) = h_3(u_3^- - D) = Q^-, \\ (1 - h - z)(u^+ - D) = (1 - h_3)(u_3^+ - D) = Q^+, \\ \frac{1}{2}(u^- - D)^2 + h + z + p^+ = \frac{1}{2}(u_3^- - D)^2 + h_3 + p_3^+ = J^-, \\ \frac{1}{2}(u^+ - D)^2 + p^+ = \frac{1}{2}(u_3^+ - D)^2 + p_3^+ = J^+. \end{gathered} \tag{5.1}$$

A continuous solution of (5.1), relating the subcritical and supercritical states 3 and 4, is possible only when the condition for critical flow over the top of the obstacle is satisfied:

$$\Delta_c = \frac{(u_c^- - D)^2}{h_c} + \frac{(u_c^+ - D)^2}{1 - h_c - z_c} - 1 = 0. \tag{5.2}$$

In this case $\Delta_3 < 0$ in front of the obstacle and $\Delta_4 > 0$ behind it. For a given value $\delta = z_{max}$ the conditions (5.1) and (5.2) give an additional relation between h_3 and γ_3, which is necessary for a unique construction of the solution of the mixed problem in the region in front of the obstacle. The supercriticality of state 4 behind the obstacle means that $\lambda^{\pm}_4 < D$ and small perturbations from region I do not reach the obstacle. In region I, however, the corresponding self-similar solution may contain discontinuities that propagate with a velocity greater than D and the flow becomes inconsistent. In this case the obstacle is immersed and cannot control the upstream flow, whereby a different (completely subcritical) streamline flow regime occurs and the solution of the problem (3.1), (3.2) accords with the self-similar solution of the same problem above an even bottom, considered in Section 4. Since the conditions for obstacle immersion can be determined only after constructing the solution in region II in front of the obstacle, subsequent analysis will be devoted to a study of the set of allowable values of the problem parameters, for which the give obstacle controls the upstream flow.

The (D, δ) Plane. The obstacle is characterized by the dimensionless parameters D and $\delta = z_{max}$. It is convenient, therefore, to represent the possible configurations of waves, arising in region II, in the parameter plane (D, δ). The pattern thus obtained depends on the initial state (h_2, γ_2). The wave configuration in front of the obstacle has been studied in detail in [3] for $\gamma_2 = 0$ (both layers are at rest in the unperturbed state). The diagram of flow regimes obtained in [3], however, is substantially more complicated than that given below (compare Fig. 8 of [3] with Fig. 5). This is because different laws of conservation are used for the flow of a two-layer liquid over an obstacle and over an even bottom. Approximate laws of conservation of horizontal momentum in each layer are used as the relations at discontinuities above an even bottom while the total pressure in each layer is conserved in steady motions over an obstacle. The region of parameters on the (D, δ) plane (the (F_{10}, H) plane in [3]) for which the streamline flow over the obstacle is completely supercritical and the region in which obstacle controls the upstream flow, therefore, intersect and hysteresis becomes possible, i.e., the pattern of streamline flow for the values of (D, δ) from the region of intersection can depend on how the body is accelerated as it enters the steady-state mode. For Eqs. (3.1) the boundaries of the above-mentioned regions agree and hysteresis due to inconsistency of the complete laws of conservation of the momentum and energy does not appear. The dependence on the history of the motion of the body for a fixed streamline flow regime, which is considered below, does persist, however.

To construct (in the (D, δ) plane) a region in which for given initial values of (h_2, γ_2) the obstacle controls the upstream flow it is sufficient to consider a wave of any amplitude, propagating to the right, and to find for it all the values of the parameters (D, δ) for which the obstacle sustains it. We obtain the desired region by varying all of the allowable values for the amplitude of waves propagating to the right. Suppose that $\gamma = \gamma^+(h; h_2, \gamma_2)$ is the equation of a wave adiabatic curve of the second kind. If the value of h_3 from the region of definition of that adiabatic curve is given, then the state (h_3, γ_3), $\gamma_3 = \gamma^+(h_3; h_2, \gamma_2)$ behind the front of that wave is given (Fig. 3b).

The state (h_3, γ_3) is subcritical relative to an obstacle moving with velocity D, upon satisfaction of the inequality

$$\lambda^-(h_3, \gamma_3) < D < \lambda^+(h_3, \gamma_3).$$

The dependence z = z(h) stemming from Eqs. (5.1) above the obstacle is illustrated in Fig. 4a. Since the value of h_3 corresponds to the subcritical state, we can prove that at that point dz/dh < 0 and, therefore, the equation z(h) = 0 has two more roots h_4, h_5 such that $h_4 < h_3 < h_5$, $(dz/dh)|_{h = h_4} > 0$, and, therefore, state 4 is supercritical. The continuous solution of Eqs. (5.1) can join only states 3 and 4, provided that the parameter δ is chosen so that $\delta = \max_{h \in [h_4, h_3]} z(h)$ (Fig. 4a). To substantiate this statement, it is sufficient to note that by virtue of (5.1) the dependence z = z(h) is given by the equation

$$P(h, z) = \frac{1}{2}\frac{(Q^-)^2}{h^2} - \frac{1}{2}\frac{(Q^+)^2}{(1 - h - z)^2} + h + z - J_- + J_+ = 0$$

and the derivative dz/dh is found from

$$\left(1 - \frac{(Q^+)^2}{(1 - h - z)^3}\right)\frac{dz}{dh} = \frac{(Q^-)^2}{h^3} + \frac{(Q^+)^2}{(1 - h - z)^3} - 1 = \Delta.$$

Since $\Delta|_{h = h_3} < 0$, for a root h_4 such that $h_4 < h_4$ the signs of Δ and dz/dh for $h = h_4$ agree and state 4 is supercritical.

When only a stable discontinuity ($\lambda^+(h_3, \gamma_3) > \sigma(h_3, \gamma_3; h_2, \gamma_2)$) propagates to the right an additional constraint consists in the condition $D < \sigma(h_3, \gamma_3; h_2, \gamma_2)$, which ensures that the configuration depicted in Fig. 3b can exist.

The region ARQ′PE of parameters in the (D, δ) plane, for which the flow to the right of the obstacle is controlled by the obstacle, is shown in Fig. 5a for the initial state $h_3 = 0.26$, $\gamma_3 = -0.8$. The configuration agrees qualitatively with the corresponding diagram in [3, Fig. 8]. The curve PQ corresponds to those values of (D, δ) for which the lower layer ($D = u_3^-$ $\delta = h_3$), is completely blocked and above this curve the top of the barrier is in the upper layer. A wave propagating up along the stream thus no longer depends on the height of the obstacle. The equation of the curve PQ is given in terms of the wave adiabatic curve $\gamma = \gamma^+(h; h_2, \gamma_2)$ as follows (in a coordinate system such that the total discharge $hu^- + (1 - h)u^+ = 0$):

$$D = u_3^- = (1 - \delta)\gamma^+(\delta; h_2, \gamma_2).$$

The region ABCE corresponds to the values of the parameters of a stable discontinuity (internal surge), moving with velocity $\sigma > D$. A dashed line represents a set of pairs (D, δ), corresponding to the fixed state (h_3, γ_3) behind the discontinuity. From the stability conditions (3.6) for $D < D_{max}$ the inequality $\sigma < \lambda^+(h_3, \gamma_3)$ is satisfied, so that on the limiting curve AE the velocity of the obstacle agrees with discontinuity velocity σ = D. The dependence z = z(h) is the same above the obstacle as inside the region (Fig. 4a). For points of the (D, δ) plane above the line EC, which corresponds to the values of (h_3, γ_3) for a surge with maximum amplitude and velocity $\sigma = D_{max}$, the wave propagating to the right consists of the surge of maximum amplitude and a simple wave adjacent to it. Therefore, $D < \lambda^+(h_3, \gamma_3)$ is a constraint. Equality is reached at the line PE. In this case the flow is critical in front of the obstacle, the dependence z = z(h) above the obstacle is shown in Fig. 4b. The region ABQ corresponds to waves with $h_3 < h_2$. This region is missing in [3] since there the problem solved is analogous to the problem (3.1), (3.2), but with the special initial data $h_1 = h_2$, $\gamma_1 = \gamma_2 = 0$, and the solution of the system (3.1) in region I does not admit the existence of flow with a detached rarefaction wave. The obstacle is immersed and an entirely subcritical flow regime is realized in the region ABS.

The same main flow regimes, during which the obstacle controls the upstream flow, as noted in [3], i.e., regimes 3C and 4C, are thus possible in the region AQPE. Thus, regime 3C with a stable discontinuity corresponds to the region ABCE and regime 4C, in which the discontinuity is followed by a centered S wave, corresponds to the region CPE.

When the initial depth h_2 of the lower layer increases the region ABCE extends in front of the obstacle because the maximum amplitude of the stable shock waves decreases. This region is absent for $h_2 = 0.5$, $\gamma_0 = 0$ (Fig. 5b). As h_2 increases further stable discontinuities are possible only for $h_3 < h_2$ (sinking waves) and the region ABCE occupies the position indicated in Fig. 5c for $h_0 = 0.8$, $\gamma_0 = 0$.

For sufficiently low values of h_3 the flow subcriticality condition $\lambda^-(h_3, \gamma_3) < D < \lambda^+(h_3, \gamma_3)$ comes into contradiction with the condition of complete blocking of the lower layer $h_3 = \delta$, $u_3^- = D$ on the curve PQ. The line R'Q' with $D = \lambda^-(h_3, \gamma_3)$, bounding the region of states ARQ'PE, for which the obstacle controls the upstream flow, thus arises (Fig. 5a).

The initial shift of velocity $\gamma_2 \neq 0$ does not fundamentally change the location of the respective regions in the (D, δ) plane, although some features of flow in the vicinity of the obstacle do become more pronounced. The nonmonotonic dependence z = z(h) above the obstacle for $D > D_{max}$ leads to an abrupt rearrangement of the streamline flow pattern for small changes in the flow parameters and to hysteresis. These topics are discussed below in the examination of critical and supercritical regimes of streamline flow (regimes E and F [3]).

6. Critical Regime of Streamline Flow. As mentioned above, for the values of (D, δ) outside the region AEPQ (Fig. 5a) of self-similar solutions such that the obstacle controls the upstream flow a critical regime does not exist and, generally speaking, supercritical streamline flow should occur. Experiments ([3], regimes 4E, 5E) show, however, that in the region PEF critical flow in front of the obstacle is converted into supercritical flow behind it. The flow is supercritical above the obstacle and a hydraulic jump occurs behind it, reducing the depth of the lower layer. To explain why such a configuration arises, we consider the motion of an obstacle with velocity $D_M < D_{max}$ (Fig. 5a). For a given height δ the corresponding state in the (D, δ) plane is mapped by a point on the straight line MN. If this point lies below the curve AE, the streamline flow regime is supercritical. In the region ABPE the detached wave is controlled by the obstacle (regimes 3C and 4C [3]). At the point G of intersection of the curve PE with the straight line MN the maximum possible wave amplitude is reached and the flow in front of the obstacle becomes critical, i.e., $D = \lambda^+(h_3, \gamma_3)$. In the (h, z) plane the curve z = z(h) touches the horizontal axis at the point h_3 (Fig. 4b) and δ is equal to the value of the local maximum z_0 of that curve. A further increase in the height δ does not cause the state (h_3, γ_3) to change. The parameters of the flow above the front slope of the obstacle corresponds to the motion of the point (h, z) along the supercritical curve 3 to the level $z = \delta$.

In principle, on the reverse slope the solution can return to the initial point along the same curve, i.e., a completely supercritical regime of supercritical flow. Because of the local maximum on the curve z = z(h), however, a transition from the supercritical branch 3 to the supercritical branch 1 in a hydraulic jump is possible in the stationary solution. The only jump that satisfies the stability conditions (3.6) is a transition from branch 3 to the critical state corresponding to the maximum on the curve z = z(h).

Experiments and computations of unsteady state show that precisely this regime of streamline flow is realized. Strictly speaking, it cannot be assumed to be steady since $\Delta(z_0) = 0$ and by virtue of (5.1) derivative solutions are not limited in the neighborhood of the critical point $x = x_c$ above the reverse slope ($z'(x_c) > 0$. This can cause the position of the jump to oscillate relative to the obstacle; the amplitude of the oscillations should decrease for more extended obstacles and the jump can be considered to be quasisteady. Within the framework of the steady-state flow pattern, of course, preference cannot be given to one of the solutions mentioned above and an adequate choice can be made with allowance for the initial unsteady stage when flow enters the self-similar regime. Associated with this circumstance is the phenomenon of hysteresis, which can be realized during the motion of an obstacle of height $\delta = z_0$, depending on whether or not because of the initial conditions there is a subcritical regime of streamline flow past the obstacle along branch 2 or a supercritical regime along branch 3. Similar matters stemming from the nonunique nature of the streamline flow regimes are discussed in greater detail in the following sections for the motion of a body with velocity $D > D_{max}$.

7. Supercritical Streamline Flow. The velocity of propagation of perturbations has a maximum D_{max} in the two-layer system under consideration. The flow in front of the obstacle, therefore, is unperturbed and since $D_{max} > \lambda^+(h_2, \gamma_2)$ only a supercritical streamline flow regime for which the states in front of and behind the obstacle coincide can be supercritical relative to the moving body. Above the body, however, the flow pattern can be very interesting and this case deserves more attention. At velocities D slightly greater than D_{max} the hydraulic jump of maximum amplitude "drives into" the body and stop at a certain height on the front slope. The corresponding dependence $z = z(h)$ is shown in Fig. 4c. Branches 1 and 3 are supercritical and 2 is subcritical. For an obstacle of height $\delta < z_0$ the flow is symmetric about the body and is supercritical. For an obstacle of height $\delta > z_0$ a continuous solution does not exist since branch 2 corresponds to subcritical flow and continuous transition from branch 1 to branch 3 is not possible on either the front or reverse slopes of the obstacle. In much the same way as for the case considered in Section 6 there is a discontinuous solution that contains a pair of jumps, changing the flow from supercritical to critical. The pattern of the streamline flow in the (h, z) plane is shown in Fig. 4c. The arrows to the curves correspond to flow on the front slope and those to the left, to flow on the reverse slope of the obstacle. On the front slope the supercritical flow rises to a height z_1, then comes a hydraulic jump taking the flow in the critical state to branch 3, followed by supercritical flow along branch 3 to the value $z = \delta$. On the reverse slope at a level z_0 a jump takes the flow to branch 1, along which it returns to the initial state 2. Also applicable to the solution constructed are the comments from Section 6 about the unlimited nature of the derivatives at the levels $z = z_0$ and $z = z_1$ and the quasisteady nature of the jumps. From the standpoint of control of the flow during the motion of the body such a flow regime should be classified as supercritical since the states of the flow in front of and behind the obstacle coincide. In much the same way as for the critical streamline flow regime considered above, when the height of the obstacle is increased the flow regimes change abruptly in the passage through the value $z_{max} = z_0$ and when the height decreases, in the passage through the value $z_{max} = z_1$. The hysteresis obtained is more pronounced than for the case of critical streamline flow and can result in a nonunique streamline flow pattern as well as abrupt unsteady changes in the flow during smooth variation of the problem parameters. As the velocity D of the obstacle increases further the dependence $z = z(h)$ becomes monotonic and the flow, symmetric. The possibility of reaching various asymptotic forms of the solution for long times can be studied on the basis of a numerical simulation of the nonstationary problem. Section 8 gives the results of numerical computations concerning mainly the nonuniqueness of the regime of steady-state streamline flow past an obstacle.

8. Unsteady Flow of a Two-Layer Liquid. A number of assumptions were made in the analysis of self-similar solutions of the problem of a streamline flow of a two-layer liquid past a body moving with velocity D. The main assumptions are that: the flow enters the self-similar regimes fairly rapidly, the flow over the obstacle is steady, the condition arises for the flow to advance to the body in the layer in which the body moves ($u^- < D$), and the body controls the upstream flow in the normal and forced streamline flow regimes. The feasibility of each assumption can be ascertained from the asymptotic form of the solutions of the nonstationary problem. Direct analysis is hindered because the equations are nonlinear, but numerical computation makes it possible to verify or refute the various hypotheses.

On the basis of a conservative difference scheme for Eqs. (2.1) with the laws of conservation (3.1) we solved the problem of motion of an obstacle in a two-layer liquid along the horizontal bottom. For various methods of accelerating the body in the initial segment to reach a steady towing regime we used all the main flow regimes found experimentally and theoretically from an analysis of self-similar solutions, in particular the arrival at the critical and supercritical regimes of streamline flow in the appropriate region of problem parameters. We used the scheme of straight-through computations without isolating the discontinuities, analogous to the scheme of Godunov, cited in [13]. The stability condition (3.6), therefore, cannot be used to construct a numerical algorithm, but the principle that discontinuities are stable against expansion into a series of waves of smaller amplitude is equivalent to (3.6) and is intrinsic to such difference schemes led to a natural selection of only stable discontinuities in the numerical solution.

Of particular interest is the numerical computation of forced flow regimes corresponding to critical and supercritical streamline flow past an obstacle, since the flow over an obstacle has a very fine structure and, strictly speaking, is not steady. The computations show that in this case as well the wave pattern of the flow rapidly reaches asymptotic form that is stable against perturbations of the oncoming stream. Figure 6a shows the pattern of streamline flow past an obstacle for values of the parameters (D, δ) from the region PEF, corresponding to the critical regime E4 from [3]. The continuous sinking wave is on the reverse slope of the obstacle and the state of the flow is critical in front of the obstacle and supercritical above the obstacle; this corresponds to a stationary solution, which is considered above and takes state 3 into 4 (Fig. 4b). The calculation of a supercritical streamline flow past an obstacle, containing a jump–whirl pair, is shown in Fig. 6b. The flow regime is most distinct for the values $h_0 \sim 0.26$, $\Lambda_0 \sim -0.8$, $D = 0.36$, $\delta = 0.2$, i.e., experimentally it is more easily detected when the obstacle is towed in a two-layer system with an initial shift of velocity in the layers. The last solution is a beautiful test for checking the numerical algorithm, since it represents supercritical asymmetric flow with a large gradient, for which the boundary conditions ca be easily formulated.

As mentioned above, the stationary solution is not determined uniquely for values of the obstacle height δ that lie in the interval (z_1, z_0). For $\delta = z_0$ the solution on the leading edge of the wave can correspond to the subcritical branch 2 or the supercritical branches 1, 3 of the curve $z = z(h)$ (Fig. 4b, c). Figure 7 shows the pattern of streamline flow past an triangular obstacle moving with velocity $D = 0.35$ in a flow with an initial velocity shift $\gamma_0 = -0.8$ and lower-layer depth $h_0 = 0.26$, for the case $\delta = z_0 = 0.15$. Rectilinear generatrices of the body are more convenient for determining the location of the jumps above the obstacle. The values of the determining parameters are chosen so that the towing regime corresponds to point E in the (D, δ) plane (Fig. 5a). Since for this point $D = D_{max}$ and $\delta = z_0$, three different asymptotic limits in the problem of unsteady acceleration of a body and its reaching a steady towing regime can be found from the solution of the self-similar problem. All three flow regimes can be simulated in numerical computation. Supercritical symmetric flow (dashed line in Fig. 7), corresponding to curve 1 in Fig. 4b, is established when the barrier is accelerated rapidly (dimensionless time $t \sim 5$) to a velocity $D = D_{max}$. In the case of smooth acceleration ($t \sim 300$) a detached hydraulic jump manages to form in front of the body and when the body reaches the velocity $D = D_{max}$ the jump "sits" on the obstacle (dash-and-dot line in Fig. 7); on the front slope the subcritical flow corresponds to curve 2 and supercritical flow on the reverse slope, to curve 1 (Fig. 4b).

Numerical simulation of asymmetric supercritical flow above a barrier containing a jump–whirl pair is somewhat more complicated. For this purpose at first an obstacle of height $\delta \sim 0.4$ accelerates smoothly ($t \sim 100$) so that a regime of forces streamline flow, corresponding to the region PEF in Fig. 5 (regime 4E [3]). The height of the obstacle then smoothly decreases ($t \sim 100$) to $\delta = 0.15$. The stable streamline flow regime obtained is indicated by the solid line in Fig. 7.

At least theoretically, therefore, three different asymptotic limits for long times have been constructed for the general set of motion parameters (h_2, γ_2; D, δ) in the problem of supercritical streamline flow of a two-layer liquid past a body. An interesting aspect here is the possible change in the type of flow when the determining parameters of the motion change only slightly, with the change in flow leading to a considerable rearrangement of the flow as a whole. In particular, in the transition from a forced to a supercritical regime of streamline flow the energy accumulated above the body can be studied downstream along the flow in the form of waves of large amplitude.

9. Remarks. One of the main problems in the study of two-layer flows in the Boussinesq approximation is that of choosing the laws of conservation to reflect the evolution of the internal hydraulic jumps. The difficulty in simulating such flows is that the condition of potential flow in each layer in the transition to the long-wave approximation contradicts the law of conservation of momentum. This contradiction can be eliminated within the complete model that takes into account the possibility of part of the energy of long waves being pumped into short-wave perturbations at the interface, with an attendant growth of the interlayer between layers [11]. When the interlayer thickness is relatively small, however, the indicated system reduces to the equations of two-layer shallow water (2.1) with the laws of conservation (3.1), based on the equations of the continuity and potentiality of the flow in each layer. And although this system is approximate, as shown above it gives quantitative as well as qualitative agreement with the results of experiments on the structure of the waves in the problem of unsteady streamline flow of a two-layer liquid past a body.

For long times the solution of this problem tends to the self-similar solution of the problem of the decay of an arbitrary discontinuity with auxiliary boundary conditions on the line $x = Dt$ in the (x, t) plane, corresponding to the trajectory of the motion of an obstacle towed with velocity D.

The conditions above the obstacle that relate the states of rest of the flow in front of and behind the body make it possible to determine the flow everywhere by solving the mixed problem in the region in front of the obstacle and the Cauchy problem in the region behind it. These conditions are determined by the possible existence of a flow that is steady relative to the obstacle, effecting a subcritical–supercritical transition. In contrast to one-layer shallow water, however, they do not necessarily conform to the generally accepted "control cross section" conditions, under which there is a continuous steady flow over the obstacle, subcritical on the front slope of the barrier and supercritical on its reverse slope. Forced streamline flow regimes, for which the flow is supercritical above the crest of the obstacle arise because of the characteristic features of the propagation of long-wave perturbations in the two-layer liquid. The fact that the nonlinear waves have a maximum velocity D_{max} causes the plane of the parameters (D, δ) to have a region (PEF in Fig. 5a) where a change in the height δ of the towed body cannot have an upstream effect but at the same time the flow above the obstacle is asymmetric, since the streamline flow regime, which contains a hydraulic jump (to be more exact, a well) taking the flow from branch 3 to branch 1 (Fig. 4b), is more stable. This last circumstance causes hysteresis, i.e., the dependence of the pattern of streamline flow past the body on the history of its entry into the steady-state regime. Numerical computations of unsteady motion in a two-layer liquid, as well as experiments [3] confirm that asymptotic flow in the neighborhood of an obstacle is not unique for the determining towing parameters from region PEF (Fig. 5a). When the body moves with a velocity greater than D_{max}, an asymmetric streamline flow regime, containing a pair of hydraulic jumps (a jump on the front slope and a well on the reverse slope, Fig. 6b), is also possible. The most interesting here (especially from the standpoint of application in oceanology and meteorology) is the possibility of the streamline flow regime "breaking down" to a symmetric regime, which leads to large unsteady changes in the flow in the neighborhood of the immersed body for small changes in the parameters of the oncoming stream.

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